Constructing initial data for the Einstein equations

Justin Corvino

Lafayette College
Easton, PA, U.S.A.

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Overview

Goals

- Introduce and motivate the Einstein Constraint Equations.
- Explore the role of the constraints in the initial-value problem for Einstein's equation, and thus appreciate why it is of interest to construct solutions to the constraint equations.
- Introduce and explore approaches for solving the Einstein Constraint Equations, including conformal and non-conformal techniques, and gluing constructions.

Outline

Part I: Introducing the Einstein constraint equations.

- Derive the Einstein equation from the Einstein-Hilbert action.
- Derive the Einstein constraint equations.
- Solutions to the constraints as initial data for the Einstein equation.

Part II: Constructing initial data: conformal techniques.

- Conformal deformations of scalar curvature.
- The basic framework of the conformal method of Lichnerowicz, Choquet-Bruhat, York: attempt to parametrize the moduli space of solutions.
- CMC case.
- Gluing via conformal deformations.

Outline

Part III. Localized deformation.

- Fischer-Marsden analysis. Localized deformation of scalar curvature.
- Localization of CSC gluing: conformal and localized techniques

Part IV. Deformation and gluing with an approximate kernel.

- Gluing and asymptotics for asymptotically flat initial data
- N-body construction

Epilogue (as time permits)

- Constraints operator and its linearization. Killing initial data (KID).
- Hamiltonian formulation of the Einstein equation: the constraints operator and energy-momentum of isolated systems.
- Survey of deformation, gluing and asymptotics for the full constraints operator.

Curvature conventions

Obligatory curvature convention slide.

We define the curvature for an affine connection ∇ as $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

We use the index conventions (and the Einstein summation convention)

$$R = R^{\ell}_{ijk} \frac{\partial}{\partial x^{\ell}} \otimes dx^i \otimes dx^j \otimes dx^k$$

and with metric $g = \langle \cdot, \cdot \rangle$, $R_{ijk\ell} = \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}), \frac{\partial}{\partial x^{\ell}} \rangle = R^m_{ijk} g_{m\ell}$.

Ricci curvature: $R_{jk} = R^{\ell}_{\ell jk} = R_{kj}$.

Scalar curvature: $R(g) = g^{ij} R_{ij}$.

Derivatives: Comma denotes partial derivative, semi-colon denotes covariant derivative. $\nabla g = 0$, i.e. $g_{ij;k} = 0$, $g^{ij}_{\ ;k} = 0$.

The Einstein Equation

The vacuum Einstein equation for $(\mathcal{S}, \bar{g} = \langle \cdot, \cdot \rangle)$ is $\mathrm{Ric}(\bar{g}) = 0$. This equation characterizes stationary points of the Einstein-Hilbert action

$$\bar{g} \mapsto \int_{\mathcal{S}} R(\bar{g})\, dv_{\bar{g}}.$$

Indeed the linearization $L_{\bar{g}} := DR_{\bar{g}}$ of the scalar curvature map $\bar{g} \mapsto R(\bar{g})$ is given by $L_{\bar{g}}\bar{h} := \frac{d}{dt}\Big|_{t=0} R(\bar{g} + t\bar{h})$, and

$$\begin{aligned} L_{\bar{g}}\bar{h} &= -\bar{g}^{ij}\bar{g}^{k\ell}\bar{h}_{k\ell;ij} + \bar{g}^{ki}\bar{g}^{\ell j}\bar{h}_{k\ell;ij} - \bar{g}^{ki}\bar{g}^{\ell j}\bar{h}_{k\ell}\bar{R}_{ij} \\ &= -\Delta_{\bar{g}}(\mathrm{tr}_{\bar{g}}\bar{h}) + \mathrm{div}_{\bar{g}}\mathrm{div}_{\bar{g}}\bar{h} - \langle \bar{h}, \mathrm{Ric}(\bar{g})\rangle_{\bar{g}} \end{aligned}$$

while the linearization of the volume element $\bar{g} \mapsto dv_{\bar{g}} = \sqrt{|\det(\bar{g}_{ij})|}\, dx$ is given by

$$\frac{1}{2}\bar{g}^{k\ell}\bar{h}_{k\ell}\, dv_{\bar{g}} = \frac{1}{2}\mathrm{tr}_{\bar{g}}\bar{h} dv_{\bar{g}} = \frac{1}{2}\langle \bar{h}, \bar{g}\rangle_{\bar{g}} dv_{\bar{g}}.$$

The Einstein Equation

Thus for $\bar{h}$ compactly supported (away from boundary if $\partial\mathcal{S}$ nonempty) the variation of the Einstein-Hilbert action is

$$\int_{\mathcal{S}} -\langle \bar{h},\ \mathrm{Ric}(\bar{g}) - \tfrac{1}{2}R(\bar{g})\bar{g}\rangle_{\bar{g}}\, dv_{\bar{g}} =: \int_{\mathcal{S}} -\langle \bar{h},\ G(\bar{g})\rangle_{\bar{g}}\, dv_{\bar{g}}$$

where $G(\bar{g}) = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g}$ is the Einstein tensor.
Remark: The Einstein tensor is divergence-free (Bianchi identity).

Stationary metrics for the Einstein-Hilbert action are those which satisfy $G(\bar{g}) = 0$, which in case $\dim(\mathcal{S}) \geq 3$ reduces to $\mathrm{Ric}(\bar{g}) = 0$ (upon tracing $G(\bar{g}) = 0$ to get $R(\bar{g}) = 0$).

For Λ constant, stationary points of the action $\bar{g} \mapsto \int_{\mathcal{S}} (R(\bar{g}) - 2\Lambda)\, dv_{\bar{g}}$ are those for which $G_\Lambda(\bar{g}) := \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} + \Lambda\bar{g} = 0$. In dimension $\dim(\mathcal{S}) = 4$, we get $\mathrm{Ric}(\bar{g}) = \Lambda\bar{g}$.

Variational characterization of CSC

Quick detour: (M, g) closed Riemannian, $\dim(M) \geq 3$.
$[g] = \{fg : f : M \to (0, +\infty)\}$ is the conformal class of g.
Einstein-Hilbert action: $\mathcal{R}(g) = \int_M R(g)\, dv_g$.

Volume-normalized action: $\overline{\mathcal{R}}(g) := \frac{\mathcal{R}(g)}{(\mathrm{Vol}(g))^{\frac{n-2}{n}}} = \mathcal{R}((\mathrm{Vol}(g))^{-\frac{2}{n}}\, g)$,
where $\bar{g} = (\mathrm{Vol}(g))^{-\frac{2}{n}}\, g \in \mathcal{M}_1$ (unit volume metrics).

Critical equations

- g is critical for $\mathcal{R}$ iff $\mathrm{Ric}(g) = 0$ (Vacuum Einstein)
- g is critical for $\overline{\mathcal{R}}$ iff $\mathrm{Ric}(g) = \frac{R(g)}{n}\, g$ (Einstein, implies CSC), equivalently $g \in \mathcal{M}_1$ critical for $\mathcal{R}|_{\mathcal{M}_1}$ iff $g \in \mathcal{M}_1$ is Einstein
- g critical for $\overline{\mathcal{R}}|_{[g]}$ iff $R(g)$ is constant (CSC), equivalently, $g \in \mathcal{M}_1$ critical for $\mathcal{R}|_{\mathcal{M}_1 \cap [g]}$ iff $R(g)$ is constant.

The Einstein Equation

Finally, we note that in case of matter fields Ψ, say, we consider a modified Lagrangian action with matter terms modeled by Lagrangian density $\mathcal{L}_m = \hat{\mathcal{L}}_m \sqrt{|\det \bar{g}|}$, where κ is a constant ($\kappa = \frac{8\pi G}{c^4}$ in space-time dimension four) :

$$(\bar{g}, \Psi) \mapsto \int\limits_{\mathcal{S}} \left[\frac{1}{2\kappa}(R(\bar{g}) - 2\Lambda) + \hat{\mathcal{L}}_m(\bar{g}, \Psi)\right] dv_{\bar{g}}$$

The energy-momentum tensor T is defined so that the variation of the action with respect to (compactly supported) metric variations of the form $\bar{g} + \varepsilon \bar{h}$ yields

$$\int\limits_{\mathcal{S}} -\frac{1}{2\kappa}\left[\langle \bar{h}, G_\Lambda(\bar{g}) - \kappa T\rangle_{\bar{g}}\right] dv_{\bar{g}}.$$

T is automatically divergence-free, and the Euler-Lagrange equation yields Einstein's equation:

The Einstein Equation

The Einstein Equation

$$G_\Lambda(\bar{g}) = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} + \Lambda\bar{g} = \kappa T.$$

Without a specific matter model in mind, any metric satisfies such an equation, by definition!

We often consider specific matter models (Einstein-Maxwell, e.g.), vacuum case ($T = 0$), or at least impose some energy condition on T, such as the dominant energy condition (DEC), which plays a role in the Positive Mass Theorem:

For all future-directed timelike vectors ξ, the vector $-T^a_{\ b}\xi^b$ is future-directed causal, i.e. for all timelike or causal ξ, χ in the same time-cone, $T(\xi, \chi) \geq 0$.

The Second Fundamental Form

We will study the geometry of space-like hypersurfaces M inside a space-time $(\mathcal{S}, \bar{g})$, of dimension $\dim(\mathcal{S}) = m + 1$, where we will for simplicity often take $m = 3$.

For the moment, let $M \subset (\mathcal{S}, \bar{g})$ be an embedded submanifold, where $\bar{g} = \langle \cdot, \cdot \rangle$ is non-degenerate (Riemannian or Lorentzian, say).

We assume the metric $\bar{g}$ induces a non-degenerate metric g (the First Fundamental Form) on M. For the induced metric g, the Levi-Civita connection ∇ is related to the connection $\overline{\nabla}$ of $\bar{g}$ by

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^{\mathrm{Tan}} + (\overline{\nabla}_X Y)^{\mathrm{Nor}} = \nabla_X Y + \mathbb{II}(X, Y).$$

For X and Y tangent to M, $\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$ is also tangent to M, so that $\mathbb{II}(X, Y) = \mathbb{II}(Y, X)$. From this we can then see that $\mathbb{II}(X, Y)$ is tensorial in X and Y. The symmetric tensor $\mathbb{II}$ is called the vector-valued second fundamental form.

The Second Fundamental Form

In the case M is a hypersurface ($\dim(M) = m = \dim(\mathcal{S}) - 1$), we let n be a (local) unit normal field to M.
From $\langle \overline{\nabla}_X n, n \rangle = 0$, we see $\overline{\nabla}_X n$ is tangent to M.
We also note that for X and Y tangent to M, $\langle \overline{\nabla}_X Y, n \rangle = -\langle Y, \overline{\nabla}_X n \rangle$.

Scalar-valued second fundamental form

We define $\hat{K}$ by
$\hat{K}(X, Y)n = \mathbb{II}(X, Y) = \langle n, n \rangle \langle \mathbb{II}(X, Y), n \rangle n = \langle n, n \rangle \langle \overline{\nabla}_X Y, n \rangle n$.
We let $K(X, Y) = \langle -\overline{\nabla}_X n, Y \rangle = \langle \overline{\nabla}_X Y, n \rangle = \langle n, n \rangle \hat{K}(X, Y)$.
We refer to K or $\hat{K}$ as the scalar-valued second fundamental form.

In case $(\mathcal{S}, \bar{g})$ is Lorentzian, $\langle n, n \rangle = -1$ and so

$$\hat{K}(X, Y) = \langle \overline{\nabla}_X n, Y \rangle.$$

The Gauss Equation

There are fundamental equations related the curvature tensor $\bar{R}$ of $\bar{g}$, R of g and the second fundamental form $\mathbb{II}$ (or $\hat{K} = \langle n, n \rangle K$).
The Gauss equation relates the curvature tensor on the submanifold M to that of the ambient manifold, with the difference measured using the second fundamental form.

The Gauss Equation

For X, Y, Z, W tangent to M,

$$\begin{aligned} \langle R(X, Y, Z), W \rangle = \langle \bar{R}(X, Y, Z), W \rangle \\ + \langle \mathbb{II}(X, W), \mathbb{II}(Y, Z) \rangle - \langle \mathbb{II}(X, Z), \mathbb{II}(Y, W) \rangle \end{aligned}$$

Hypersurface case:

$$\begin{aligned} \langle R(X, Y, Z), W \rangle = \langle \bar{R}(X, Y, Z), W \rangle \\ + \langle n, n \rangle \Big[K(X, W) K(Y, Z) - K(X, Z) K(Y, W) \Big] \end{aligned}$$

The Gauss Equation

Proof: The proof proceeds by decomposing terms in $\bar{R}(X,Y,Z) = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$ into tangential and normal components, like $\overline{\nabla}_X Y = \nabla_X Y + \mathbb{II}(X,Y)$:

$$\begin{aligned}\langle \bar{R}(X,Y,Z), W\rangle &= \langle \overline{\nabla}_X(\nabla_Y Z + \mathbb{II}(Y,Z)), W\rangle \\ &\quad - \langle \overline{\nabla}_Y(\nabla_X Z + \mathbb{II}(X,Z)), W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle\end{aligned}$$

and then again for $\overline{\nabla}_X \nabla_Y Z$ (similarly for the other term)

$$\overline{\nabla}_X(\nabla_Y Z + \mathbb{II}(Y,Z)) = \nabla_X \nabla_Y Z + \mathbb{II}(X, \nabla_Y Z) + \overline{\nabla}_X(\mathbb{II}(Y,Z)).$$

Upon inner product with W (tangent to M), we obtain

$$\begin{aligned}\langle \overline{\nabla}_X(\nabla_Y Z + \mathbb{II}(Y,Z)), W\rangle &= \langle \nabla_X \nabla_Y Z, W\rangle + \langle \overline{\nabla}_X(\mathbb{II}(Y,Z)), W\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \mathbb{II}(Y,Z), \overline{\nabla}_X W\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \mathbb{II}(Y,Z), \mathbb{II}(X,W)\rangle.\end{aligned}$$

The Codazzi Equation

The Codazzi equation involves the normal component $\bar{R}^{\perp}(X, Y, Z)$ of $\bar{R}(X, Y, Z)$, whereas the Gauss equation involved the tangential component of this vector. We need only the hypersurface case:

The Codazzi Equation (hypersurface case)

For X, Y, Z tangent to M,

$$\langle \bar{R}(X, Y, Z), n\rangle = \langle n, n\rangle\Big((\nabla_X \hat{K})(Y, Z) - (\nabla_Y \hat{K})(X, Z)\Big).$$

Gauss and Codazzi in Index notation

Let i, j, k and ℓ be indices for components tangential to M, and the n index indicates the vector n is placed in the indicated slot of the tensor.

Gauss:

$$R_{ijk\ell} = \bar{R}_{ijk\ell} + \langle n, n\rangle(\hat{K}_{i\ell}\hat{K}_{jk} - \hat{K}_{ik}\hat{K}_{j\ell}) = \bar{R}_{ijk\ell} + \langle n, n\rangle(K_{i\ell}K_{jk} - K_{ik}K_{j\ell})$$

Codazzi:

$$\bar{R}_{ijkn} = \langle n, n\rangle(\hat{K}_{jk;i} - \hat{K}_{ik;j}) = K_{jk;i} - K_{ik;j}$$

The Einstein Constraint Equations

Consider a hypersurface M in a space-time $(\mathcal{S}, \bar{g})$, so that the induced metric g on M is Riemannian. Let n be a (local) time-like unit normal field, which we take to be future-pointing.

We suppose that $\bar{g}$ satisfies an Einstein equation

$$G_\Lambda(\bar{g}) = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} + \Lambda\bar{g} = \kappa T.$$

The Einstein constraint equations relate the first and second fundamental forms g and K on M.

They are obtained using the Gauss and Codazzi equations, together with the information about the ambient curvature contained in the Einstein equation.

The Einstein Constraint Equations: DEC

Let $\tilde{\mathbb{J}}^\nu = -T^{\mu\nu} n_\mu = -T^{\mu\nu} n^\beta \bar{g}_{\mu\beta}$. We write $\tilde{\mathbb{J}} = \rho n + \vec{J}$, with $\langle \vec{J}, n\rangle = 0$.
We let J be the one-form dual to $\vec{J}$ using the metric g on M.
If we let $E_1, \dots, E_m$ be a basis for $T_p M$, with dual basis $\theta^1, \dots, \theta^m$, then we can write $\vec{J} = J^\ell E_\ell$ and $J = J_i \theta^i$, with $J_i = J^\ell g_{i\ell}$. We let $E_0 = n$ to give a basis $\{E_\mu\}$ of $T_p \mathcal{S}$.

It's not hard to see (watch minus signs! E.g. $n^0 = 1$, $\bar{g}_{00} = -1$, $n_0 = -1$)

Dominant Energy Condition

$\rho = T(n,n)$; this is the energy density of matter fields as measured by an observed with four-velocity cn.

J is ($\pm c$ times) the corresponding observed momentum density one-form: $J^i = -T^{\mu i} n_\mu = T^{0i}$, and $J_j = g_{j\ell} J^\ell = \bar{g}_{j\mu} T^{0\mu} = -T_{0j} = -T_{j0}$.

Dominant energy condition: $\tilde{\mathbb{J}}$ is future-pointing causal:

$$\rho \geq |J|_g = \sqrt{J^i J_i}$$

The Einstein Constraint Equations

The Einstein Constraint Equations

$$R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 = 2\kappa\rho + 2\Lambda$$

$$\mathrm{div}_g(K - (\mathrm{tr}_g K)g) = \mathrm{div}_g K - d(\mathrm{tr}_g(K)) = \kappa J.$$

In case $m = 3$, say, this is locally a system of four equations for (g, K), forming an underdetermined elliptic system.

Let $\Lambda = 0$ for now. Vacuum case would be $\rho = 0$, $J = 0$.

Maximal case: $\mathrm{tr}_g(K) = 0$: $R(g) = 2\kappa\rho + |K|_g^2 \geq 0$ (under DEC).

Time-symmetric case: $K = 0$: $R(g) = 2\kappa\rho \geq 0$ (under DEC) .

In this case, the scalar curvature is proportional to the observed local energy density.

Time-symmetric vacuum case: $R(g) = 0$.

We give the proof of the first constraint, *the Hamiltonian constraint*.

Proof: Let E_i be an orthonormal frame for T_pM. We use the Gauss equation (careful with the signs: $\langle n, n\rangle = -1$):

$$\sum_{i,j=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i\rangle = \sum_{i,j=1}^{m} \Big[\langle R(E_i, E_j, E_j), E_i\rangle + K(E_j, E_j)K(E_i, E_i) - (K(E_i, E_j))^2\Big]$$
$$= R(g) - |K|_g^2 + (\mathrm{tr}_g(K))^2.$$

Again, $\langle n, n\rangle = -1$, so that

$\overline{\mathrm{Ric}}(E_j, E_j) = -\langle \bar{R}(n, E_j, E_j), n\rangle + \sum_{i=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i\rangle$. Thus

$$\sum_{i,j=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i\rangle = \overline{\mathrm{Ric}}(n, n) + \sum_{j=1}^{m} \overline{\mathrm{Ric}}(E_j, E_j).$$

Deriving the Einstein Constraint Equations

Recall the Einstein tensor: $G = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g}$.

$$\begin{aligned}
R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= \sum_{i,j=1}^{m} \langle \bar{R}(E_i, E_j, E_j), E_i \rangle \\
&= \overline{\mathrm{Ric}}(n,n) + \sum_{j=1}^{m} \overline{\mathrm{Ric}}(E_j, E_j). \\
&= 2\overline{\mathrm{Ric}}(n,n) + (-\overline{\mathrm{Ric}}(n,n) + \sum_{j=1}^{m} \overline{\mathrm{Ric}}(E_j, E_j)) \\
&= 2\overline{\mathrm{Ric}}(n,n) + R(\bar{g}) \\
&= 2G(n,n) \\
&= 2(-\Lambda\bar{g} + \kappa T)(n,n) \\
&= 2\Lambda + 2\kappa\rho
\end{aligned}$$

Deriving the Einstein Constraint Equations

As for the momentum constraint, we employ the Codazzi equation, which we recall in index form: $\bar{R}_{ijkn} = K_{jk;i} - K_{ik;j}$.

Proof: Since $\bar{R}_{jinn} = 0$, we have
$G_{in} = \bar{R}_{in} = \sum_{j=1}^{m} \langle \bar{R}(e_j, e_i, n), e_j \rangle = -\sum_{j=1}^{m} \bar{R}_{jijn} = -\sum_{j=1}^{m} (K_{ij;j} - K_{jj;i})$. Now, by the Einstein equation, $G_{in} = \kappa T_{in} = -\kappa J_i$, which finishes the proof.

Constraints and components of the Einstein tensor

$$2\Lambda + 2\kappa\rho = 2G_{nn} = R(g) - |K|_g^2 + (\mathrm{tr}_g(K))^2.$$

$$-\kappa J_i = G_{in} = -(\mathrm{div}_g(K) - d(\mathrm{tr}_g(K))_i = -\mathrm{div}_g(K - (\mathrm{tr}_g(K))g)_i.$$

There are lots of solutions of the constraint equations

There are lots of solutions to the Einstein Constraint Equations. In fact, if we don't restrict T in any way, then any space-like hypersurface in *any* Lorentzian manifold will satisfy the constraints.

If we impose restrictions on T, like the dominant energy condition, or the vacuum condition $T = 0$, then that restricts the geometry of $(\mathcal{S}, \bar{g})$ in some way, but still, of course, any Riemannian hypersurface yields a solution to the constraints.

Keep in mind the constraints form an underdetermined system, so in fact we might expect there to be a wide variety of solutions. In fact, from the point of view of the initial value formulation, this variety is useful for attempting to model a variety of physical situations.

Example: Minkowski space-time

Example

$(\mathcal{S}, \bar{g}) = (\mathbb{R}^{m+1}, \eta)$, Minkowski space-time, $\eta = -dt^2 + \sum_{i=1}^{m} (dx^i)^2$.

- $M = \{t = 0\}$. With the induced metric g, $(M, g) \cong (\mathbb{R}^m, g_{\mathbb{E}})$ and clearly $\overline{\nabla}_X n = 0$, so that $K = 0$. The constraints are trivial to verify.
- $M = \{-t^2 + |x|^2 = -1\}$. (M, g) is congruent to hyperbolic space $\mathbb{H}^m \cong (\mathbb{R}^m, g_{\mathbb{H}})$ of curvature -1. Moreover, since $n = x^\mu \frac{\partial}{\partial x^\mu}$ $(x^0 := t)$, for Y tangent to M, $\overline{\nabla}_Y n = Y$, so $K = -g$. One can easily verify the vacuum constraints.

Remark: There are constant-curvature examples for $\Lambda > 0$ (DeSitter space-time) and $\Lambda < 0$ (anti-DeSitter space-time).

Example: Schwarzschild

If one imposes rotational symmetry on a solution to the Einstein vacuum equation $\mathrm{Ric}(\bar{g}) = 0$, one obtains the following solutions ($G = 1$, $c = 1$, $m = 3$), with coordinates (t, x^i), $\tilde{r} = |x|$, and $\mathring{g}_{\mathbb{S}^2} = d\Omega^2$ is the round unit sphere metric:

The Schwarzschild solution

$$\bar{g}_S = -\left(1 - \frac{2m}{\tilde{r}}\right) dt^2 + \left(1 - \frac{2m}{\tilde{r}}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 \mathring{g}_{\mathbb{S}^2}.$$

m is a constant of integration in one of the second-order ODE obtained; a second constant has been re-scaled to 1.

m is called the *mass*.

Note that the space-time is asymptotically Minkowksian as $\tilde{r} \to +\infty$.

Example: Schwarzschild

Example (Schwarzschild)

$M = \{t = 0\}$ with induced metric $g_S = (1 - \frac{2m}{\tilde{r}})^{-1} d\tilde{r}^2 + \tilde{r}^2 \mathring{g}_{\mathbb{S}^2}$.

g_S must solve the vacuum Einstein constraint equations.

In fact, the second fundamental form vanishes:
$\langle \overline{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial t} \rangle = \langle \overline{\Gamma}^0_{ij} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = 0$. (Easy exercise)

So $K = 0$ (time-symmetric), and the vacuum constraint equations reduce to the vanishing of the scalar curvature: $R(g_S) = 0$.

Example: Schwarzschild

We can re-write g_S in a convenient form, by making a radial change of coordinate, using a positive increasing function r of $\tilde{r}$ to write $g_S = (1 - \frac{2m}{\tilde{r}})^{-1} d\tilde{r}^2 + \tilde{r}^2 \mathring{g}_{\mathbb{S}^2} = (u(r))^4 (dr^2 + r^2 \mathring{g}_{\mathbb{S}^2})$:

$g_S = (1 - \frac{2m}{\tilde{r}})^{-1} \left(\frac{d\tilde{r}}{dr}\right)^2 dr^2 + \left(\frac{\tilde{r}}{r}\right)^2 r^2 \mathring{g}_{\mathbb{S}^2}$.

We arrange

$$\left(1 - \frac{2m}{\tilde{r}}\right)^{-1/2} \frac{d\tilde{r}}{dr} = \frac{\tilde{r}}{r} =: (u(r))^2$$

so that we can factor out of the metric g_S to obtain the proposed *conformally flat* representation of g_S.

We can solve the ODE (by separation, and the substitution $\tilde{r} = m + m\cosh w$, and imposing $\frac{\tilde{r}}{r} \to 1$ as $\tilde{r} \to +\infty$) to get $\frac{\tilde{r}}{r} = \left(1 + \frac{m}{2r}\right)^2$.

Example: Schwarzschild

We thus obtain the following form of the Schwarzschild metric:

$$\bar{g}_S = -\frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2} dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 \mathring{g}_{\mathbb{S}^2}\right)$$

$g_S = \left(1 + \frac{m}{2r}\right)^4 g_{\mathbb{E}}$ is conformally flat, with $R(g_S) = 0$, which corresponds to the fact that $\left(1 + \frac{m}{2|x|}\right)$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

There is a higher-dimensional analogue too!

Example (Schwarzschild $m > 0$)

$(\mathbb{R}^n \setminus \{0\}, g_S, K = 0)$ with

$$g_S = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/(n-2)} g_{\mathbb{E}^n}.$$

Exercise: What if $m < 0$? Compare the area profile $A(r) := \mathrm{Area}(\Sigma_r)$, where $\Sigma_r = \{|x| = r\}$ in the case $m > 0$ to the case $m < 0$.

Initial Value Formulation

Note that the constraints come from imposing $G_\Lambda(\bar{g})(n,\cdot) = \kappa T(n,\cdot)$, and as we saw above, this does not involve time derivatives of the space-time metric, but can be expressed in terms of g and K on the slice. This is qualitatively similar to Maxwell's equations, where the divergence equations must be satisfied at $t = 0$; this restricts the allowable initial data for the electromagnetic fields.

The Einstein constraint equations are necessary for (M, g, K) to be a space-like slice of a space-time $(\mathcal{S}, \bar{g})$ satisfying the Einstein equation. Question: Are the equations also sufficient, say in the vacuum case, or with suitable matter models? We remark that we are not imposing the full Gauss and Codazzi equations; in fact, we haven't imposed the space-time $(\mathcal{S}, \bar{g})$ into which we are embedding (M, g, K) either!

Answer: Yes! Y. Choquet-Bruhat, 1952. (M, g, K) can be interpreted as initial data for a Cauchy problem for Einstein's equation, which in suitable coordinates say, can be represented as a nonlinear hyperbolic system.

Initial Value Formulation

We will only illustrate the basic ideas of the foundational work of Choquet-Bruhat; to be mathematically precise, we should specify function spaces and state carefully the relevant partial differential equations results. In order to formulate the initial value problem as a nonlinear wave equation, we express the Einstein equations in terms of a partial differential equation *along with* a gauge condition.

Wave coordinates

The gauge choice we use is the choice of coordinates. In fact, we will use *harmonic coordinates*, or in the Lorentzian case, *wave coordinates* x^α, which are coordinates so that the gauge functions $\lambda^\alpha := \Box_{\bar{g}} x^\alpha = 0$.

On any Lorentzian manifold we can locally set up wave coordinates: given any local coordinates y^μ, we solve the Cauchy problem for the linear wave equations $\Box_{\bar{g}} x^\mu = 0$ with initial conditions $x^\mu = y^\mu$ and $\nabla_n x^\mu = \nabla_n y^\mu$ impose on a level surface of y^0 ($\frac{\partial}{\partial y^0}$ is time-like), where n is the unit normal to the level surface.

Initial Value Formulation

Outline

- PDE. Reduced Einstein equation: Einstein equation in wave coordinates, quasilinear hyperbolic system.
- Initial data. At $t = 0$:
  Let $\bar{g}_{00} = -1$, $\bar{g}_{0j} = 0$, $j \geq 1$.
  (g, K) determine $\bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu,0}$ for $\mu, \nu \geq 1$.
  Remaining data: $\bar{g}_{\mu 0,0}$.
- Constraints. At $t = 0$, $G_{0\mu} = 0$ (or $(G_\Lambda)_{0\mu} = 0$ at $t = 0$).
  We know we these must play a role!

Remark: Note that while we said above that given a Lorentzian $\bar{g}$ we can find wave coordinates, we will be building the metric from initial data, solving a system that gives us what we want if we are in wave coordinates for $\bar{g}$ after constructing it!

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$\Gamma^k_{ij} = \frac{1}{2}\bar{g}^{km}(\bar{g}_{im,j} + \bar{g}_{jm,i} - \bar{g}_{ij,m})$ be the Christoffel symbols for $\bar{g}$ in a coordinate system. (Indices are space-time indices unless noted.)
It's easy to see

$$\lambda^\alpha = \Box_{\bar{g}} x^\alpha = \bar{g}^{ij} x^\alpha_{;ij} = \bar{g}^{ij}(-\Gamma^k_{ij} x^\alpha_{,k}) = -\bar{g}^{ij}\Gamma^\alpha_{ij}.$$

In what follows we will write "$A \sim B$" to mean $A - B$ is a function of the components $\bar{g}_{ij}$ and $\bar{g}_{ij,k}$; in particular, $A - B$ does not depend on second derivatives of the metric components. For example,

$$-(\bar{g}_{\alpha i}\lambda^\alpha_{\ ,j} + \bar{g}_{\alpha j}\lambda^\alpha_{\ ,i}) \sim \bar{g}_{\alpha i}\bar{g}^{km}\Gamma^\alpha_{km,j} + \bar{g}_{\alpha j}\bar{g}^{km}\Gamma^\alpha_{km,i}.$$

Now the components of the Ricci curvature of $\bar{g}$ are given by

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik} \sim \Gamma^k_{ij,k} - \Gamma^k_{ik,j}.$$

Initial Value Formulation

We can put this together with a straightforward calculation:

Reduced Ricci

- Gauge terms: $\frac{1}{2}(\bar{g}_{\alpha i}\lambda^{\alpha}_{\ ,j} + \bar{g}_{\alpha j}\lambda^{\alpha}_{\ ,i}) \sim -\frac{1}{2}\bar{g}^{km}(\bar{g}_{ki,mj} + \bar{g}_{jm,ki} - \bar{g}_{km,ij})$.
- Leading order Ricci terms:

$$
\begin{aligned}
R_{ij} &\sim \Gamma^k_{ij,k} - \Gamma^k_{ik,j} \\
&\sim \frac{1}{2}\bar{g}^{km}[(\bar{g}_{im,jk} + \bar{g}_{jm,ik} - \bar{g}_{ij,mk}) - (\bar{g}_{im,kj} + \bar{g}_{km,ij} - \bar{g}_{ik,mj})] \\
&= \frac{1}{2}\bar{g}^{km}(\bar{g}_{jm,ik} - \bar{g}_{ij,mk} - \bar{g}_{km,ij} + \bar{g}_{ik,mj}).
\end{aligned}
$$

- Reduced Ricci: $R^H_{ij} := R_{ij} + \frac{1}{2}(\bar{g}_{\alpha i}\lambda^{\alpha}_{\ ,j} + \bar{g}_{\alpha j}\lambda^{\alpha}_{\ ,i}) \sim -\frac{1}{2}\bar{g}^{mk}\bar{g}_{ij,mk}$

The reduced vacuum Einstein equation $R^H_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$ can thus be written as a quasi-linear wave equation:

Initial Value Formulation

The reduced vacuum Einstein equation $R^H_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$:

$$-\frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}_{\mu\nu,\alpha\beta} + \Psi_{\mu\nu}((\bar{g}_{ij}), (\bar{g}_{ij,k})) = 0. \tag{1}$$

A solution to (1) will solve the Einstein equation if we can arrange $\lambda^\alpha = 0$ for all $\alpha \geq 0$. From the seminal work of Leray, along with a rescaling argument, we know we can solve a system like (1) for small time.

Choose local coordinates x^i, $i \geq 1$ on $U \subset M$. We will construct a local solution on the product of an interval (the $x^0 = t$ interval) and U.

Prescribing Initial data. Suppose we are given a solution (M, g, K) of the Einstein constraint equations for $\mathrm{Ric}(\bar{g}) = \Lambda \bar{g}$. We prescribe the metric components and first derivatives at $t = 0$:

We let $\bar{g}_{00} = -1$ and for $\mu \geq 1$, $\bar{g}_{0\mu} = 0$. Then for $\mu, \nu \geq 1$, we use the data coming from the solution of the constraints: $\bar{g}_{\mu\nu} = g_{\mu\nu}$, and $\bar{g}_{\mu\nu,0} = \langle \overline{\nabla}_{\partial_0}\partial_\mu, \partial_\nu\rangle + \langle \partial_\nu, \overline{\nabla}_{\partial_0}\partial_\nu\rangle = -2K_{\mu\nu}$. For $\mu \geq 0$, we will choose the initial values of $\bar{g}_{0\mu,0}$ to arrange the gauge condition, as we explain.

Initial Value Formulation

Outline

- Gauge condition at $t = 0$. Arrange the gauge condition at $t = 0$ by choosing remaining data $\bar{g}_{\mu 0,0}$ at $t = 0$.
- Solve the reduced system. Solve the reduced Einstein equation for $\bar{g}_{\mu\nu}$ with given initial conditions $\bar{g}_{\mu\nu}|_{t=0}$ and $\bar{g}_{\mu\nu,0}|_{t=0}$.
  This will solve the vacuum Einstein equation provided the gauge condition holds for all t.
- Time derivative of gauge condition at $t = 0$. Constraints imply the initial time derivative of the gauge condition vanishes.
- PDE for gauge functions. The gauge functions satisfy a second-order evolution equation (Bianchi identity plus reduced Einstein). Since the Cauchy data vanish, the gauge functions vanish identically.
- Conclusion: We have a solution of the vacuum Einstein equation.

Initial Value Formulation

Remarks on remaining calculations

- Expand $\lambda^\alpha = -\bar{g}^{ij}\Gamma^\alpha_{ij}$ at $t=0$. Spatial derivatives of metric are known, show that the time derivatives $\bar{g}_{\mu 0,0}$ can be chosen to arrange $\lambda^\alpha = 0$ at $t = 0$.
- Assuming $R^H_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$, show $(G_\Lambda(\bar{g}))_{\mu\nu} = -\frac{1}{2}\bar{g}_{\alpha\mu}\lambda^\alpha_{\ ,\nu} - \frac{1}{2}\bar{g}_{\alpha\nu}\lambda^\alpha_{\ ,\mu} + \frac{1}{2}\bar{g}_{\mu\nu}\lambda^\alpha_{\ ,\alpha}$
- By the constraints, $(G_\Lambda(\bar{g}))_{\mu 0} = 0$ at $t = 0$. Expand the above at $t = 0$ to conclude $\lambda^\alpha_{,0} = 0$ at $t = 0$ too.
- Use the Bianchi identity $(\mathrm{div}_{\bar{g}}(G_\Lambda(\bar{g})) = 0$ and the form of $(G_\Lambda(\bar{g}))_{\mu\nu}$ for the solution of the Reduced Einstein equation to derive a linear hyperbolic system (principal term $\Box_{\bar{g}}\lambda^\alpha$) for the λ^α with vanishing Cauchy data!

Solving the constraints

Discussion

- We see that producing interesting solutions to the constraints should produce interesting space-times.
- We'd like to have ways to generate solutions to the vacuum constraints (apart from taking slices in known space-times).
- We remark that the constraint system is undetermined, and we have lots of freedom.
- But the system is nonlinear and is non-trivial.
- Is there a way to effectively parametrize the space of solutions?

Let's start with the time-symmetric vacuum constraint $R(g) = 0$ on $\mathbb{R}^n$ for now...

Solving the constraints

Consider the vanishing scalar curvature $R(g) = 0$ equation on $\mathbb{R}^n$.
Let $g_{\mathbb{E}}$ be the Euclidean metric on $\mathbb{R}^n$.
A simple attempt to construct more solutions: try $g_{\mathbb{E}} + h$, where h has small norm. Could we take h compactly supported too?

Spoiler alert

If h were compactly supported, then $g = g_{\mathbb{E}} + h$ can't be a nontrivial solution...by the PMT.

For example, you could generate a "trivial" such solution by considering a nontrivial diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$, so that outside a compact set, F is the identity. Then we can see $F^*(g_{\mathbb{E}}) = g_{\mathbb{E}} + h$.

Solving the constraints

- So, we do not produce anything new with $R(g_{\mathbb{E}} + h) = 0$ for h compactly supported. (Same for $R(g_{\mathbb{E}} + h) = 0 \geq 0$.)
- Note, however, that the linearized equation $L_{g_{\mathbb{E}}}(h) = -\Delta_{g_{\mathbb{E}}}(\mathrm{tr}_{g_{\mathbb{E}}} h) + \mathrm{div}_{g_{\mathbb{E}}}(\mathrm{div}_{g_{\mathbb{E}}} h)$ does admit nontrivial compactly supported solutions!
- Exercise: Show that there are nontrivial compactly supported symmetric tensors h on $\mathbb{R}^3$ which are divergence-free and trace-free (with respect to $g_{\mathbb{E}}$). Such tensors are called transverse-traceless (TT).
- Such TT tensors h represent tensor directions transverse to directions generated by diffeomorphisms (if X is a vector field generating a one-parameter family of diffeomorphisms φ_t, then $L_X g = X_{i;j} + X_{j;i} = \frac{d}{dt}\big|_{t=0} \varphi_t^* g$).

Solving the constraints

Now what? Well, we look for a conformal factor $u > 0$ to try to help us solve $R(g) = 0$ with $g = u^{4/(n-2)}(g_{\mathbb{E}} + h)$.

To represent an isolated system, we want $u \to 1$ at infinity.

For Riemannian manifolds (M, g) of dimension $n \geq 3$, we have the following formula for the scalar curvature under a conformal change:

$$R(u^{4/(n-2)}g) = -\tfrac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}}(\Delta_g u - \tfrac{n-2}{4(n-1)} R(g)u).$$

Case $n = 3$: $R(u^4 g) = -8u^{-5}(\Delta_g u - \frac{1}{8} R(g)u)$.

Example: $R(u^{\frac{4}{n-2}} g_{\mathbb{E}}) = 0$ is equivalent to $\Delta u = 0$ (u is harmonic), while $R(u^{\frac{4}{n-2}} g_{\mathbb{E}}) \geq 0$ is equivalent to $\Delta u \leq 0$ (u is *superharmonic*).

Remark: Trying to solve for u globally on $\mathbb{R}^n$ in the preceding example, with $u \to 1$ at infinity, only gives $u = 1$.

Solving the constraints

Scalar curvature under a conformal change

$$R(u^{4/(n-2)}g) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} (\Delta_g u - \frac{n-2}{4(n-1)} R(g) u).$$

- Prescribing the scalar curvature of the conformal metric gives a semi-linear elliptic equation; prescribing that it vanish gives the linear equation $\Delta_g u - \frac{n-2}{4(n-1)} R(g) u = 0$.
- Note that we want a positive solution $u > 0$.
- So it is important to understand the behavior of operators of the form $(\Delta_g - f)$, such as the conformal Laplacian $\mathcal{L}_g = \Delta_g - \frac{n-2}{4(n-1)} R(g)$.

Solving the constraints

Solving for $u > 0$, $u \to 1$ at infinity

• If we consider $g_h = g_{\mathbb{E}} + h$, $R(u^{4/(n-2)}(g_{\mathbb{E}} + h)) = 0$, for h compactly supported and small, the equation we have to solve is linear:

$$\mathcal{L}_{g_h}(u) = \Delta_{g_h} u - \frac{n-2}{4(n-1)} R(g_h) u = 0.$$

• We want $u > 0$ on $\mathbb{R}^n$, and we want $u \to 1$ at infinity.

• So let $u = 1 + v$. We want v to decay to 0 at infinity, and we set up function spaces to capture the decay of v.

• We re-write the PDE as $\Delta_{g_h} v - \frac{n-2}{4(n-1)} R(g_h) v = \frac{n-2}{4(n-1)} R(g_h)$.

Solving the constraints

Solving for $u > 0$, $u \to 1$ at infinity

- To solve: $\mathcal{L}_{g_h} v = \Delta_{g_h} v - \frac{n-2}{4(n-1)} R(g_h) v = \frac{n-2}{4(n-1)} R(g_h)$.

- The operator $\mathcal{L}_{g_h} = \Delta_{g_h} - \frac{n-2}{4(n-1)} R(g_h)$ agrees with $\Delta = \Delta_{g_{\mathbb{E}}}$ outside a compact set. For small h, $\mathcal{L}_{g_h}$ is a small perturbation of Δ, which is an isomorphism on appropriate weighted spaces. (Bounded harmonic functions are constant, and harmonic functions outside a compact set which decay to 0 at infinity decay like $O(|x|^{-(n-2)})$.)
Thus $\mathcal{L}_{g_h}$ is an isomorphism, and we can solve $\mathcal{L}_{g_h} v = \frac{n-2}{4(n-1)} R(g_h)$, for v.

- Because $R(g_h)$ is small, v will be small and tending to 0 at infinity.

- Such metrics as we've just constructed are called harmonically flat: outside a compact set, the manifold is a neighborhood of infinity in $\mathbb{R}^n$, with coordinates $x = (x^i)$ for the end in which $g_{ij} = u^{4/(n-2)} \delta_{ij}$ ($g = u^{4/(n-2)} g_{\mathbb{E}}$), $u \to 1$ at infinity, and $R(g) = 0$, i.e. $\Delta u = 0$.

Harmonically flat expansion

Spherical harmonic expansion

• Such u admits a spherical harmonic expansion:

$$u(x) = 1 + \frac{A}{|x|^{n-2}} + \frac{\beta \cdot x}{|x|^n} + \cdots$$

where A is a constant scalar and β is a constant vector.

Note also that derivatives fall off one order faster: $\frac{\partial |x|^{-1}}{\partial x^i} = -|x|^{-3} x^i$, etc.

• Such metrics are then asymptotic to Schwarzschild of mass $m = 2A$: $g - g_S = O_*(|x|^{-(n-1)})$. (Asterisk denotes derivative fall off.)

• If $A \neq 0$, we define $c = \beta/[(n-2)A]$: a straightforward computation shows

$$\tilde{u}(y) := u(y + c) = 1 + \frac{A}{|y|^{n-2}} + O(|y|^{-n}).$$

It makes sense then to identify c as the *center of mass*.

Harmonically flat expansion

• Starting with $u > 0$ defined on $\mathbb{R}^n$ so that $g = u^{4/(n-2)}(g_{\mathbb{E}} + h)$ has $R(g) = 0$, with h of compact support, then g is harmonically flat at infinity, and

$$u(x) = 1 + \frac{A}{|x|^{n-2}} + O_*(|x|^{-(n-1)}).$$

• We will hear more about the mass, but for now, we identified $m = 2A$ by comparison with Schwarzschild.

• With $g_h = g_{\mathbb{E}} + h$, we integrate

$$\Delta_{g_h} u = \frac{n-2}{4(n-1)} R(g_h) u$$

and apply the Divergence Theorem for large r ($d\sigma$ is just the surface measure on the sphere) to get...

Harmonically flat expansion

$$\int_{\mathbb{R}^n} \frac{n-2}{4(n-1)} R(g_h) u \, dv_{g_h} = \int_{\mathbb{R}^n} \Delta_{g_h} u \, dv_{g_h}$$
$$= \lim_{r\to\infty} \int_{|x|=r} \frac{\partial u}{\partial r} d\sigma$$
$$= \lim_{r\to\infty} \int_{|x|=r} -\tfrac{(n-2)A}{|x|^{n-1}} \, d\sigma$$
$$= -(n-2)A\omega_{n-1}$$

(ω_{n-1} is the area of the round unit sphere in $\mathbb{R}^n$), or

$$m = 2A = -\tfrac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} R(g_{\mathbb{E}} + h) \, u \, dv_{g_h}.$$

Remark: That $m \geq 0$ is certainly not immediately clear!!

Solving the constraints

• We will hear more about harmonically flat metrics in upcoming lectures.
• One might ask about the structure of the space of solutions to the constraints.
• Classic results of Fischer and Marsden (cf. Bartnik, Chruściel-Delay) establish certain manifold structures on the space of solutions to the constraints.
• On the other hand, it has been only very recently shown by F. C. Marques that the space of asymptotically flat metrics of zero scalar curvature on $\mathbb{R}^3$ (and other topologies too) is connected, using Ricci flow with surgery.

• A natural question is whether the space of solutions to the constraints, on a closed manifold, say, admits an effective parametrization. An attempt at this is the conformal method of Lichnerowicz, Choquet-Bruhat, York, O'Murchadha.

The Conformal Method

Overview

• The ECE forms a system of 4 equations for 12 unknowns (locally, in dimension three).

• One might try to isolate certain parts of the data as "free" data—freely specifiable, giving parameters for the space of solutions to the constraints.

• The ECE will then give a system of PDE to determine the other parts of the data (g, K).

• This system of PDE will be a determined elliptic system.

The Conformal Method

The first step is the TT decomposition of symmetric two-tensors on (M^n, g). We consider M closed (compact without boundary), and note one can set up spaces to facilitate the upcoming analysis in certain noncompact spaces (e.g. AE/AF (asymptotically Euclidean/flat)).
We want to write a symmetric two-tensor Ψ as

$$\Psi = \Psi^{TT} + \Psi^{L} + \Psi^{Tr}$$

where Ψ^{TT} is TT (divergence-free, trace-free w.r.t. g), and Ψ^{L} is trace-free, and Ψ^{Tr} is a pure trace term:

$$\Psi^{Tr} = \frac{1}{n}(\mathrm{tr}_g \Psi) g.$$

Ansatz for Ψ^{L} (note that by design Ψ^{L} is trace-free):

$$\Psi^{L}_{ab} = (L_g W)_{ab} = W_{a;b} + W_{b;a} - \frac{2}{n}(\mathrm{div}_g W) g_{ab}.$$

L_g is the conformal Killing operator; vectors in the kernel (CKV's) generate conformal isometries.

The Conformal Method

$$\Psi = \Psi^{TT} + \Psi^{L} + \Psi^{Tr}$$

Therefore for any vector field W used to define $\Psi^L = L_g W$, the tensor

$$\Psi^{TT} := \Psi - \Psi^{L} - \Psi^{Tr}$$

is symmetric and trace-free.
To deserve the "TT" superscript, then, we choose W to arrange $\mathrm{div}_g(\Psi^{TT}) = 0$, obtaining an equation for W:

$$\mathrm{div}_g(L_g W) = \mathrm{div}_g(\Psi - \Psi^{Tr}).$$

Proposition

Given Ψ, we can solve the above equation for W, uniquely up to the addition of a CKV. Thus $\Psi^L = L_g W$ is uniquely determined.

The Conformal Method

Proof: Let $\mathcal{P}_g = \text{div}_g \circ L_g$ as an operator on vector fields to vector fields:

$$\begin{aligned}(\mathcal{P}_g W)^a &= g^{ad}(L_g W)_{db;c} g^{bc} \\ &= g^{ad} W_{d;bc} g^{bc} + g^{ad} W_{b;dc} g^{bc} - \frac{2}{n} g^{ad} W^{\ell}{}_{;\ell c} g_{db} g^{bc} \\ &= W^a{}_{;bc} g^{bc} + \frac{1}{n} g^{ad} (\text{div}_g W)_{;d}\end{aligned}$$

$\mathcal{P}_g$ is elliptic: the principal symbol is obtained (think Fourier transform) by replacing derivatives by $\xi \in T_p^* M$, and is a linear operator on $T_p M$ for each $p \in M$ (up to a constant factor):

$$V \mapsto V^a \xi_b \xi_c g^{bc} + \frac{1}{n} V^{\ell} \xi_{\ell} \xi_c g^{ac} = |\xi|_g^2 V + \frac{1}{n} \xi(V) \xi^{\sharp}$$

Notice: if $\xi \neq 0$, the right-side vanishes if and only if $V = -\frac{1}{n} \frac{\xi(V)}{|\xi|_g^2} \xi^{\sharp}$. This shows $\xi(V) = 0$, which implies by the previous that $V = 0$.

The Conformal Method

$\mathcal{P}_g$ is an elliptic operator from vector fields to vector fields, which is self-adjoint:

$$\begin{aligned}\langle Z, \mathcal{P}_g W\rangle_{L^2} &= \int_M Z^a (L_g W)_{ab;c} g^{bc}\ d\mu_g \\ &= -\int_M Z^a_{\ ;c} (L_g W)_{ab} g^{bc}\ d\mu_g \\ &= -\frac{1}{2}\int_M (Z_{a;b} + Z_{b;a})(L_g W)^{ab}\ d\mu_g \\ &= -\frac{1}{2}\int_M (L_g Z)_{ab}(L_g W)^{ab}\ d\mu_g\end{aligned}$$

where in the last line we used that $L_g W$ is trace-free, and the preceding line used symmetry. But now we observe, then

$$\langle Z, \mathcal{P}_g W\rangle_{L^2} = -\frac{1}{2}\langle L_g Z, L_g W\rangle_{L^2} = \langle \mathcal{P}_g Z, W\rangle_{L^2}.$$

The Conformal Method

We note by the preceding argument we have two identities:

$$\langle W, \mathcal{P}_g W\rangle_{L^2} = -\frac{1}{2}|L_g W|^2_{L^2}$$

and for any symmetric trace-free tensor S (such as $S = \Psi - \Psi^{Tr}$, say)

$$\langle Z, \mathrm{div}_g(S)\rangle_{L^2} = -\frac{1}{2}\langle L_g Z, S\rangle_{L^2}.$$

- By the Fredholm Alternative/Hodge Decomposition, the range of $\mathcal{P}_g$ is the L^2-orthogonal complement of $\ker(\mathcal{P}_g^*) = \ker(\mathcal{P}_g)$.
- By the first identity above, the kernel of $\mathcal{P}_g$ is composed of CKV's.
- By the second identity, $\mathrm{div}_g(S)$ is L^2-orthogonal to any CKV, for any trace-free symmetric S.
- $\mathrm{div}_g(S)$ is thus in the range of $\mathcal{P}_g$ for any trace-free symmetric S, say $S = \Psi - \Psi^{Tr}$, for example!
- Thus, we can solve, uniquely up to adding a CKV, the PDE $\mathcal{P}_g W = \mathrm{div}_g(\Psi - \Psi^{Tr})$, as desired.

The Conformal Method

Remark: Such a theorem also holds on AE spaces in suitable weighted function spaces. Note that there are no CKV which decay at infinity.

Remark: we work in dimension three for simplicity here.

Let g_0 be a metric. The conformal class $\mathcal{C} = [g_0]$ of g_0 is the set of all metrics g for which there is a function $\phi > 0$ for which $g = \phi^4 g_0$. In this case we recall the identity from last time

$$R(g) = -8\phi^{-5}\big[\Delta_{g_0}\phi - \frac{1}{8}R(g_0)\phi\big].$$

We note one more conformal identity: for a symmetric trace-free (with respect to g_0 and thus g) $(2,0)$-tensor Ξ,

$$(\mathrm{div}_g(\phi^{-10}\Xi))^a = \phi^{-10}(\mathrm{div}_{g_0}\Xi)^a.$$

The Conformal Method

With these identities in mind, we now formulate the conformal method.

Conformal method

Free data: • Conformal class $\mathcal{C}$ (say $\mathcal{C} = [g_0]$).
• g_0-TT tensor σ^{cd}
• Scalar function τ

Based on this data, and on the TT-decomposition and conformal properties, we search for solutions to the constraints in the following form:

$$g = \phi^4 g_0$$
$$K^{cd} = \phi^{-10}\left(\sigma^{cd} + (L_{g_0} W)^{cd}\right) + \frac{1}{3}\phi^{-4} g_0^{cd} \tau$$

Solve for $(\phi > 0, W)$.

The Conformal Method

Conformal method

$$g = \phi^4 g_0$$

$$K^{cd} = \phi^{-10}\left(\sigma^{cd} + (L_{g_0} W)^{cd}\right) + \frac{1}{3}\phi^{-4} g_0^{cd}\tau$$

We immediately compute the momentum constraint operator, noting $\mathrm{tr}_g K = \tau$:

$$\begin{aligned}(\mathrm{div}_g K - d(\mathrm{tr}_g K))^a &= \phi^{-10}[\mathrm{div}_{g_0}(\sigma + L_{g_0} W)]^a + \frac{1}{3} g^{ab}(d\tau)_b - g^{ab}(d\tau)_b \\ &= \phi^{-10}[\mathrm{div}_{g_0}(L_{g_0} W)]^a - \frac{2}{3} g^{ab}(d\tau)_b\end{aligned}$$

Furthermore, we find $|K|_g^2 = \phi^{-12}|\sigma + L_{g_0} W|_{g_0}^2 + \frac{1}{3}\tau^2$, so with $\mathrm{tr}_g K = \tau$ and $R(g) = -8\phi^{-5}\left[\Delta_{g_0}\phi - \frac{1}{8}R(g_0)\phi\right]$, we get...

The Conformal Method

The vacuum constraints become the determined elliptic system for $\phi > 0$ and W:

Conformal reformulation of the Constraints

$$\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma + L_{g_0}W|^2_{g_0} + \frac{1}{12}\tau^2\phi^5$$

$$\mathrm{div}_{g_0}(L_{g_0}W)_a = \frac{2}{3}\phi^6 d\tau_a$$

The most successful case of the analysis of this system is the CMC case, where τ is constant. In this case the second equation is just $\mathcal{P}_{g_0}W = 0$, which implies $L_{g_0}W = 0$, and so this doesn't affect $K^{cd} = \phi^{-10}\Big(\sigma^{cd} + (L_{g_0}W)^{cd}\Big) + \frac{1}{3}\phi^{-4}g_0^{cd}\tau$ anyway, so we take $W = 0$, say.

The Conformal Method

So in the CMC case, we are left with solving a single semi-linear elliptic PDE for $\phi > 0$, the Lichnerowicz equation:

$$\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} + \frac{1}{12}\tau^2\phi^5$$

• Conformal Invariance: For (g_0, σ^{cd}, τ) (τ constant) given, for $\theta > 0$, consider $(\theta^4 g_0, \theta^{-10}\sigma^{cd}, \tau)$. (g_0, σ, τ) admits a solution $\phi > 0$ if and only if $(\theta^4 g_0, \theta^{-10}\sigma^{cd}, \tau)$ admits a solution $\phi\theta^{-1} > 0$.

• Note: In the CMC case, we saw we do not really need the TT-decomposition: If we write $K = \mathring{K} + \frac{1}{3}\tau g$, then $\mathrm{tr}_g \mathring{K} = 0$, and the momentum constraint is just $\mathrm{div}_g \mathring{K} = 0$, because τ is constant. In other words, $\mathring{K}$ is TT.
And we saw if σ^{ab} is g_0-TT, then $\phi^{-10}\sigma^{ab}$ is g-TT for $g = \phi^4 g_0$.

Theorem (Choquet-Bruhat, O'Murchadha, York, Isenberg). Let M be closed, let (g_0, σ^{cd}, τ) be CMC conformal data (τ constant). The Lichnerowicz equation admits a *positive* solution $\phi > 0$ as indicated in the table below; "$\mathcal{Y}$" indicates the Yamabe class of g_0. The solution is unique except in the case $(\mathcal{Y}^0, \sigma = 0, \tau = 0)$ in which case any positive constant is a solution. If the free data are smooth, then ϕ is smooth too.

	$\sigma \equiv 0,\ \tau = 0$	$\sigma \equiv 0,\ \tau \neq 0$	$\sigma \not\equiv 0,\ \tau = 0$	$\sigma \not\equiv 0,\ \tau \neq 0$
$\mathcal{Y}^+$	No	No	Yes	Yes
$\mathcal{Y}^0$	Yes	No	No	Yes
$\mathcal{Y}^-$	No	Yes	No	Yes

Yamabe classes: The space of metrics on M can be cleaved into three equivalence classes: $\mathcal{Y}^+$ is the set of metrics conformal to a metric of positive scalar curvature, and similarly for $\mathcal{Y}^0$, $\mathcal{Y}^-$. By the resolution of the Yamabe problem (Yamabe, Aubin, Trudinger, Schoen), each metric g on M can be conformally deformed to a metric of constant scalar curvature.

By conformal invariance, one may without loss of generality, choose g_0 so that $R(g_0)$ is constant. This isn't necessary (D. Maxwell, J Hyp. DE '05)

On the proof: Isenberg (CQG '95) completed the table, using a unified super- and sub-solution method for the existence portions of the table. Basic maximum principle/integration by parts suffice for the non-existence.

Definition

Let $\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} + \frac{1}{12}\tau^2\phi^5 =: F(x,\phi)$.
A function ϕ_+ is a *super-solution* of the equation provided $\Delta_{g_0}\phi_+ \leq F(x,\phi_+)$. Similarly, a function ϕ_- is a *sub-solution* provided $\Delta_{g_0}\phi_- \geq F(x,\phi_-)$.

Theorem

If there exist sub- and super-solutions ϕ_- and ϕ_+ with $0 < \phi_- \leq \phi_+$, then there is a solution ϕ with $0 < \phi_- \leq \phi \leq \phi_+$.

The Conformal Method

Proof of existence/non-existence:

The "No" cases are easy to prove (really!).

The "Yes" cases in the first two columns are easy too.

Proof of existence/non-existence:

Lichnerowicz Equation: $\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} + \frac{1}{12}\tau^2\phi^5$:

"No" cases I: $\sigma \equiv 0$: $\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi + \frac{1}{12}\tau^2\phi^5$.

In all these cases, using $\int_M \Delta_{g_0}\phi \, dv_{g_0} = 0$, we obtain:

• If $R(g_0) > 0$, integrate the equation to show that it does not admit a positive solution $\phi > 0$.

• If $R(g_0) = 0$, and $\tau \neq 0$, integrate the equation to show it does not admit a positive solution $\phi > 0$.

• If $R(g_0) < 0$ and $\tau = 0$: integrate $\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi$ to show it does not admit a positive solution $\phi > 0$.

"No" cases II: $\sigma \not\equiv 0$, $\tau = 0$: $\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0}$.

• If $R(g_0) \leq 0$, integration of the PDE yields that no $\phi > 0$ can be a solution.

The Conformal Method

Proof of existence/non-existence:

Lichnerowicz Equation: $\Delta_{g_0}\phi = \frac{1}{8}R(g_0)\phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} + \frac{1}{12}\tau^2\phi^5$

"Yes" Cases I (first two columns)

- $R(g_0) = 0$, $\sigma \equiv 0$, $\tau = 0$: $\Delta_{g_0}\phi = 0$. Easy.

Next column:

- $R(g_0) = -8$, $\sigma \equiv 0$, $\tau \neq 0$: $\Delta_{g_0}\phi = -\phi + \frac{1}{12}\tau^2\phi^5$. This admits a positive constant solution $\phi = C$, where $C = \frac{1}{12}\tau^2 C^5$.

"Yes" Cases II (third column) This is the case that hadn't been treated by the previous (Leray-Schauder) method:

- $R(g_0) = 8$, $\sigma \not\equiv 0$, $\tau = 0$: $\Delta_{g_0}\phi = \phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0}$.

Cheap case: If σ *never* vanishes, it's easy to check that $0 < \phi_- = \sqrt[8]{\frac{1}{8}\min|\sigma|^2_{g_0}} \leq \sqrt[8]{\frac{1}{8}\max|\sigma|^2_{g_0}} = \phi_+$ give positive sub- and super-solutions.

The Conformal Method

"Yes" Cases II (third column), continued

• $R(g_0) = 8$, $\sigma \not\equiv 0$, $\tau = 0$. $\Delta_{g_0}\phi = \phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} = F(x, \phi)$

In general σ can have zeros.

• Let $\phi_+ = A = \max\left(1, \frac{1}{8}\max|\sigma|^2_{g_0}\right) \geq 1 > 0$. Check that ϕ_+ is a super-solution: $\frac{1}{8}|\sigma|^2_{g_0}\phi_+^{-7} \leq \frac{1}{8}|\sigma|^2_{g_0} \leq A = \phi_+$, so $\Delta_{g_0}\phi_+ = 0 \leq F(x, \phi_+)$.

That was easy. What about a subsolution?

• Since the self-adjoint elliptic operator $\Delta_{g_0} - 1$ has trivial kernel (multiply $\Delta_{g_0}\psi - \psi = 0$ by ψ and integrate by parts: $\int_M (-|\nabla_{g_0}\psi|^2 - \psi^2)\, dv_{g_0} = 0$), we have that $\Delta_{g_0} - 1$ is an isomorphism (in suitable spaces) by the Fredholm Alternative.

• Subsolution. Let ϕ_- be the solution of

$$(\Delta_{g_0} - 1)\phi_- = -\frac{1}{8}|\sigma|^2_{g_0}\phi_+^{-7} \geq -\phi_+.$$

Clearly (yes?) ϕ_- cannot have a negative minimum, so $\phi_- \geq 0$. By the Strong Maximum Principle, $\phi_- > 0$ (as desired!).

"Yes" Cases II (third column), continued

• Recall: Want to solve: $\Delta_{g_0}\phi = \phi - \frac{1}{8}\phi^{-7}|\sigma|^2_{g_0} = F(x,\phi)$
We have $(\Delta_{g_0} - 1)\phi_- = -\frac{1}{8}|\sigma|^2_{g_0}\phi_+^{-7} \geq -\phi_+$, i.e. $\Delta_{g_0}\phi_- \geq (\phi_- - \phi_+)$.

• Since ϕ_+ is constant, we have $\Delta_{g_0}(\phi_- - \phi_+) = \Delta_{g_0}\phi_- \geq (\phi_- - \phi_+)$. Clearly (yes!) then $(\phi_- - \phi_+)$ cannot have a positive maximum value, so that $\phi_- - \phi_+ \leq 0$, as desired.

• Check subsolution condition: Since $\phi_+ \geq \phi_- > 0$, we now see $\Delta_{g_0}\phi_- = \phi_- - \frac{1}{8}\phi_+^{-7}|\sigma|^2_{g_0} \geq \phi_- - \frac{1}{8}\phi_-^{-7}|\sigma|^2_{g_0} \geq F(x,\phi_-)$.

Exercise: Try the other cases (last column of the table). At least one is pretty easy. When you get stuck, see Isenberg's paper (CQG '95). Also see David Maxwell's paper (J Hyp. DE '05).

The Conformal Method

The non-CMC case is far from settled! Results in the "near-CMC" regime (Isenberg and Moncrief), and "not-so-near regimes" (Holst, Nagy, Tsogtgerel; Maxwell, Gicquaud-Cang Nguyen,...). There are results about the behavior of the conformal method—Maxwell, Holst-Meier, Dahl-Gicquaud, Gicquaud- Chruściel, Premoselli,...

Isenberg-Moncrief use a semi-decoupled system approach:

$$\Delta_{g_0}\phi_k = \frac{1}{8}R(g_0)\phi_k - \frac{1}{8}\phi_k^{-7}|\sigma + L_{g_0}W_k|^2_{g_0} + \frac{1}{12}\tau^2\phi_k^5$$

$$\mathrm{div}_{g_0}(L_{g_0}W_k)_a = \frac{2}{3}\phi_{k-1}^6 d\tau_a$$

Requires $|d\tau|_{g_0}$ suitably small, and no CKV's to solve the second equation.

Reference

Please see David Maxwell's pre-print arXiv:1407.1467 [gr-qc] for more references, and discussion of the conformal method, geometric and physical framework, prospects.

Solving the constraints by gluing

Overview

- Gluing methods seek to produce a new solution to the constraints from given solutions, so that the new solution contains regions resembling regions of the data from the original pieces.
- The constraints are nonlinear, so this is a nontrivial problem.
- We will see that conformal techniques can be used for gluing (e.g. Isenberg-Mazzeo-Pollack).
- On the other hand, if one wants the new solution to contain regions in which the data agrees precisely with that of the original solutions, we call this localized gluing, since the perturbations are localized away from certain regions—certain regions are shielded from deformations required to impose the constraints upon gluing pieces together.

Conformal gluing

IMP Gluing

Isenberg, Mazzeo and Pollack (IMP) developed gluing using the CMC conformal method. (Dimension three, vacuum; I-Maxwell-P for higher dimensions and with fields.)

We formulate one of their results, and then indicate some of the details from a related result for scalar curvature gluing.

Given Σ a three-manifold, which is either connected, or has two connected components. Mark p_1, p_2 on Σ, choosing a point from each component in that case.

Suppose (γ, K) solves vacuum constraints on Σ, with $\tau = \mathrm{tr}_\gamma K$ constant, and $K = \mu + \frac{1}{3}\tau g$, where $\mu = \mathring{K}$ is the trace-free part of K.

Conformal gluing

IMP gluing

• On $\Sigma^* := \Sigma \setminus \{p_1, p_2\}$, via normal coordinates near each p_j, conformally rescale (blow up) to produce asymptotically cylindrical ends.

• Can re-scale TT-tensor $\mu = \mathring{K}$ to produce CMC solution (conformal invariance) on Σ^*.

• "Far along" the cylinders, modify the metric smoothly to precisely cylindrical, and identify isometric pieces—this produces a connected sum on components, or adding a handle if Σ had one component.

• "Large parameter" $T > 0$: tracks length of neck (in conformal metric), exact cylinder in middle unit length segment.

• Metrics were modified and patched, then TT-tensors are patched, via convex combination using cutoffs. The conformal factor is patched slightly differently.

Conformal gluing

• At the end of this, we have manifold Σ_T, metric γ_T, positive function ψ_T, and tensor μ_T (built from $\mu = \mathring{K}$). The conformal rescaling produced a solution, but the modifications to glue mean we have only an approximate solution. To modify the approximate solution:

IMP gluing

• First, correct μ_T to $\widetilde{\mu}_T$ with is γ_T-TT...using the conformal Killing operator as in the TT-decomposition to solve for a small perturbation vector field term. Use a nondegeneracy condition ruling out certain CKV's to make sure you can do this.

• Solve for modified conformal factor $\tilde{\psi}_T > 0$ solving the Lichnerowicz equation.

• The final data is the conformal data from $[\gamma_T]$, $\tilde{\mu}_T$ and τ, via $\tilde{\psi}_T$: $\Gamma_T = \tilde{\psi}_T^4 \gamma_T$, $K_T^{cd} = \tilde{\psi}_T^{-10} \tilde{\mu}^c d_T + \frac{1}{3} \tau \tilde{\psi}_T^{-4} \gamma^c d_T$.

• Note: Outside the neck region, $\Gamma_T \to \gamma$ and $K_T \to K$ exponentially fast in T.

Conformal gluing

We illustrate some of the details used in IMP in discussing the following scalar curvature gluing.

Theorem (C., Eichmair, Miao), cf. (Chruściel, Isenberg, Pollack)

Let (Σ_1, g_1) and (Σ_2, g_2) be compact n-manifolds ($n \geq 3$) with nonempty boundaries, and let $p_i \in \text{int}(\Sigma_i)$ be marked points, and let U_i be neighborhoods of p_i. Assume that each metric has CSC $R(g_i) = \sigma_n$, where $\sigma_n \in \{-n(n-1), 0, n(n-1)\}$. In case $\sigma_n > 0$, we assume that the first Dirichlet eigenvalue of $((n-1)\Delta_{g_i} + \sigma_n) = (n-1)[\Delta_{g_i} + n]$ is positive.

Then there is a family $\hat{\gamma}_T$ on $\Sigma := \Sigma_1 \# \Sigma_2$ with $R(\hat{\gamma}_T) = \sigma_n$, with $\hat{\gamma}_T \to g_1 \sqcup g_2$ on $(\Sigma_1 \setminus U_1) \sqcup (\Sigma_2 \setminus U_2)$ and $\mathrm{Vol}(\Sigma, \hat{\gamma}_T) \to \mathrm{Vol}(\Sigma_1, g_1) + \mathrm{Vol}(\Sigma_2, g_2)$.

Remark: The proof works if Σ_i is closed and if either $\sigma_n < 0$, or if $\sigma_n > 0$ and $\Delta_{g_i} + n$ has positive spectrum.

CSC gluing

Sketch of proof: Let g be g_1 or g_2. $R > 0$ will be the radius of a sufficiently small ball around the chosen points. We omit subscripts. Let $\mathring{g}$ be the round unit metric on $\mathbb{S}^{n-1}$.

- Conformally blow up neighborhood of each chosen point in quasi-normal coordinates where $g_{ij}(x) = \delta_{ij} + Q_{ij}(x)$, $Q_{ij}(0) = 0 = \partial Q_{ij}(0)$, to produce an asymptotically cylindrical end:
$$r^{-2}g = r^{-2}dr^2 + \mathring{g} + r^{-2}Q,$$
where (r, θ) are spherical coordinates based on the x coordinates.
- For $T \gg 1$, let $s = -\log r + \log R - \frac{T}{2}$. This rescales $r = Re^{-s-T/2}$, with $r^{-2}dr^2 = ds^2$. Note that $r = R$ corresponds to $s = -\frac{T}{2}$. Also, let Ψ be defined so that $\Psi^{\frac{4}{n-2}}$ is r^2 near p, interpolates to 1 between $B_R(p)$ and $B_{2R}(p)$.
- Then $r^{-2}g = \Psi^{-\frac{4}{n-2}}g = ds^2 + \mathring{g} + e^{-T}e^{-2s}R^2\hat{h}$ inside $B_R(p)$, i.e. for $s > -\frac{T}{2}$. If $\tilde{g}_c = ds^2 + \mathring{g}$ is the cylindrical metric, then $\|\widetilde{\nabla}^\ell \hat{h}\|_{\tilde{g}_c}$ has T-independent bound.

Sketch of proof, continued

- Now transition smoothly from $\Psi^{-\frac{4}{n-2}} g = ds^2 + \mathring{g} + e^{-T} e^{-2s} R^2 \hat{h}$ in $(-\frac{T}{2}, -1)$ to the exact cylindrical metric $\tilde{g}_c = ds^2 + \mathring{g}$ in $(-1, -\frac{1}{2})$.
- Do the same for both ends. One one end, take s to $-s$, glue together in the exactly cylindrical part. Get a metric γ_T on $C_T = [-\frac{T}{2}, \frac{T}{2}] \times \mathbb{S}^{n-1}$ of the form $\gamma_T = ds^2 + \mathring{g} + e^{-T} \cosh(2s) \hat{h}_T$. $\hat{h}_T = 0$ on $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$ and $\|\widetilde{\nabla}^\ell \hat{h}_T\|_{\tilde{g}_c}$ bounded independent of T.
- Approximate conformal factor: Let Ψ_T essentially be built from Ψ_1 and Ψ_2 by interpolating between their sum in the neck region $[-\frac{T}{2} + 1, \frac{T}{2} - 1] \times \mathbb{S}^{n-1}$ to 1 on either end.
- Recall that each $\Psi^{\frac{4}{n-2}} = r^2$ $(\Psi = r^{\frac{n-2}{2}})$ near marked points, with $r = Re^{-s-T/2}$. When we add the two on the common piece of the cylinder, though, one of the factors has "s" replaced by "$-s$". So on $[-\frac{T}{2} + 1, \frac{T}{2} - 1] \times \mathbb{S}^{n-1}$, $\Psi_T = 2R^{\frac{n-2}{2}} e^{-\frac{n-2}{4} T} \cosh\left(\frac{n-2}{2} s\right)$.

CSC gluing

Sketch of proof, continued: Recall that γ_T on $[-\frac{T}{2}, \frac{T}{2}]$ transitions from $\Psi^{-\frac{4}{n-2}} g = r^{-2} g$ to the cylindrical metric from one manifold, then back the other way.

- Approximate solution: $\Psi_T^{\frac{4}{n-2}} \gamma_T$.
- $R(\Psi_T^{\frac{4}{n-2}} \gamma_T) = \frac{4(n-1)}{n-2} \Psi_T^{-\frac{n+2}{n-2}} \left[-\Delta_{\gamma_T} \Psi_T + \frac{n-2}{4(n-1)} R(\gamma_T) \Psi_T \right].$
- On $[-\frac{T}{2}+1, \frac{T}{2}-1] \times \mathbb{S}^{n-1}$, $\Psi_T = 2R^{\frac{n-2}{2}} e^{-\frac{n-2}{4}T} \cosh\left(\frac{n-2}{2}\, s\right)$. On $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$, the metric $\gamma_T = ds^2 + \mathring{g}$, so that $R(\gamma_T) = (n-1)(n-2)$, while $\Delta_{\gamma_T} \Psi_T = \left(\frac{n-2}{2}\right)^2 \Psi_T$. Thus $R(\Psi_T^{\frac{4}{n-2}} \gamma_T) = 0$ on $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$.
- The approximate solution is thus the two given manifolds joined by a Schwarzschild neck of mass $m_T = 2R^{n-2} e^{-\frac{(n-2)T}{2}}$. The volume is approximately $\mathrm{Vol}(g_1) + \mathrm{Vol}(g_2)$.

CSC gluing

Sketch of proof, continued

- Goal: Solve for conformal factor to modify γ_T to CSC: solve $\mathcal{N}_T(\Psi_T + \eta_T) = 0$ for small η_T, where

$$\mathcal{N}_T(f) = -\Delta_{\gamma_T} f + \tfrac{n-2}{4(n-1)} R(\gamma_T) f - \tfrac{n-2}{4(n-1)} \sigma_n f^{\frac{n+2}{n-2}}.$$

- $\mathcal{N}_T(\Psi_T)$ is exponentially small in T. A contraction mapping argument will give the solution, once the linearized operator $\mathcal{L}_T = D\mathcal{N}_T\Big|_{\Psi_T}$ is analyzed.
-

$$\mathcal{L}_T(f) = -\Delta_{\gamma_T} f + \tfrac{n-2}{4(n-1)} R(\gamma_T) f - \tfrac{n-2}{4(n-1)} \sigma_n \tfrac{n+2}{n-2} \Psi_T^{\frac{4}{(n-2)}} f.$$

- The operators $\mathcal{L}_T$ converge locally on punctured Σ_1 or Σ_2, or on the neck, as follows.

Sketch of proof, continued

$$\mathcal{L}_T(f) = -\Delta_{\gamma_T} f + \frac{n-2}{4(n-1)} R(\gamma_T) f - \frac{n-2}{4(n-1)} \sigma_n \frac{n+2}{n-2} \Psi_T^{\frac{4}{(n-2)}} f.$$

- Away from the neck, $\gamma_T = \Psi^{-\frac{4}{n-2}} g =: \tilde{g}$ for large T, and $\Psi_T \approx \Psi$.
- On punctured Σ_1 or Σ_2, then $\mathcal{L}_T$ converges to $\mathcal{L} = -\Delta_{\tilde{g}} + \frac{n-2}{4(n-1)} R(\tilde{g}) - \frac{n-2}{4(n-1)} \sigma_n \frac{n+2}{n-2} \Psi^{\frac{4}{n-2}}$
- $\mathcal{L}_T$ converges locally uniformly on the neck (down the neck Ψ_T is small) to $-\Delta_{\tilde{g}_c} + \frac{n-2}{4(n-1)} R(\tilde{g}_c) = -\Delta_{\tilde{g}_c} + \frac{(n-2)^2}{4}$.
- The linearized operator has no kernel for large T, by contradiction. If there were elements η_j of maximum value 1 in the kernel of $\mathcal{L}_{T_j}$ for $T_j \to +\infty$, then there are two cases to consider.

CSC gluing

Sketch of proof, continued

- Case (i): For infinitely many j, $\max|\eta_j|$ is uniformly bounded away from zero outside the neck. With a little work using the operator $\mathcal{L}$, one can show that a subsequence will determine a nontrivial solution $\hat{\eta} = \Psi^{-1}\eta$ of $(\Delta_g + \frac{\sigma_n}{n-1})\hat{\eta} = 0$ (Dirichlet), which cannot exist (by assumption in case $\sigma_n > 0$).
- Case (ii): If η_j instead go to zero locally uniformly on the punctured manifolds, then a subsequence converges to a nontrivial bounded solution of $(\Delta_{\tilde{g}_c} - \frac{(n-2)^2}{4})\eta = 0$ on the cylinder, which cannot exist.
- A similar argument shows that the inverse $\mathcal{G}_T$ is bounded independent of T.
- We can estimate the solution, in order to estimate the resulting volume, close to $\mathrm{Vol}(\Sigma_1, g_1) + \mathrm{Vol}(\Sigma_2, g_2)$.
- We have a family of metrics parametrized by T, with CSC σ_n, with volume approaching $\mathrm{Vol}(\Sigma_1, g_1) + \mathrm{Vol}(\Sigma_2, g_2)$ and which outside the gluing region approach $g_1 \sqcup g_2$.

Summary of CSC gluing result

Let's recall: we were discussing one way to glue together two manifolds of constant scalar curvature σ_n, using conformal techniques. The conformal class of the solution is that of γ_T, which interpolates the original metric on one piece to the cylinder (via conformal blow up and modification), and then back to the other metric. The function Ψ_T interpolates from 1 outside the gluing region, so that the approximate solution $\Psi_T^{\frac{4}{n-2}}\gamma_T$ interpolates between the original data away from the gluing region, and a Schwarzschild neck of small mass. One can solve for large T for an appropriately small perturbation η_T so that $R((\Psi_T + \eta_T)^{\frac{4}{n-2}}\gamma_T) = \sigma_n$. Away from the gluing region, η_T small, and so the solution is close to the original metrics!

Remark: The metrics produced solve the vacuum constraints (with $\Lambda = \frac{1}{2}\sigma_n$), with initial data (metric) near the original metric outside the neck.

Localized deformation and gluing

Of course, there were nondegeneracy conditions that needed to be assumed, both in the CSC and IMP (conformal method for CMC constraint gluing). For the IMP gluing, one makes an assumption ruling out certain CKV's, to make sure one can use the operator $W \mapsto \mathcal{P}W = \mathrm{div}(LW)$ to correct a certain trace-free tensor to be TT.

In the CSC gluing, there is an eigenvalue condition. For instance, in the closed case (compact empty boundary), the CSC construction cannot work in case CSC $\sigma_n = 0$, in general. For example, there is no way to connect together a flat torus to another flat torus: the connected sum does not admit a metric of zero scalar curvature!!

A natural question: can a gluing construction be carried out (and under what circumstances—nondegeneracy condition) to produce solutions to the constraints (e.g. CSC metrics) which preserves exact copies of regions in each data set in the final solution?

Fischer-Marsden Analysis

Note: If you preserve regions in the initial data, you will preserve some regions in the spacetime evolution of the data too.

To answer such questions, we study more general deformations than conformal deformations.

Consider the scalar curvature map $g \mapsto R(g)$, which we recall has linearization L_g, and formal adjoint L_g^* given by

Linearization of scalar curvature

- $L_g(h) := \frac{d}{dt}\Big|_{t=0} R(g+th) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - h \cdot \mathrm{Ric}(g)$.
- $L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\mathrm{Ric}(g)$.

Convention: $\Delta_g f = \mathrm{tr}_g(\mathrm{Hess}_g f) = g^{ij} f_{;ij} = g^{ij}[f_{,ij} - \Gamma_{ij}^k f_{,k}]$.

Note: $\int_M f \, L_g(h) \, dv_g = \int_M L_g^* f \cdot h \, dv_g = \int_M (L_g^* f)_{ij} h_{k\ell} g^{ik} g^{j\ell} \, dv_g$

Note that the principal part of $L_g L_g^*$ is $(n-1)\Delta_g^2$.

Static potential

Definition

A nontrivial element in the kernel of L_g^* is called a static potential

$L_g^* f = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\,\mathrm{Ric}(g)$ and so
$\mathrm{tr}_g(L_g^* f) = -(n-1)\Delta_g f - fR(g)$.

Example (Kernel of L_g^*)

- Euclidean space $\mathbb{R}^n$, with $L_{g_{\mathbb{E}}}^* f = 0$ if and only if $\mathrm{Hess}_{g_{\mathbb{E}}} f = 0$, so the kernel is spanned by constant and linear functions of Cartesian coordinates.
- The flat torus $\mathbb{T}^n$ with one-dimensional kernel given by constant functions.
- The round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, with basis for the kernel given by restriction of the coordinate functions $x^j\Big|_{\mathbb{S}^n}$, $j = 1, \ldots, n+1$.

Static potential

If $L_g^* N = 0$, then $\bar{g} = -N^2 dt^2 + g$ is Einstein on $\{N \neq 0\}$:
$\mathrm{Ric}(\bar{g}) = \frac{R(g)}{n-1}\bar{g}$, where $n = \dim(M)$.
In particular, for nontrivial such N, we can conclude that $R(g)$ is (locally) constant on $\{N \neq 0\}$; in fact it can be shown that $\{N = 0\}$, if nonempty, is a smooth totally geodesic hypersurface, and so $\{N \neq 0\}$ is dense in M.

Example (Schwarzschild (three space dimensions))

$$\bar{g}_S = -\frac{\left(1 - \frac{m}{2|x|}\right)^2}{\left(1 + \frac{m}{2|x|}\right)^2} dt^2 + g_S = -\frac{\left(1 - \frac{m}{2|x|}\right)^2}{\left(1 + \frac{m}{2|x|}\right)^2} dt^2 + \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{E}}$$

$N = \frac{\left(1-\frac{m}{2|x|}\right)}{\left(1+\frac{m}{2|x|}\right)}$ is in the kernel of $L_{g_S}^*$. For $m > 0$, $\{N = 0\} = \{|x| = \frac{m}{2}\}$, which is the *minimal sphere*.

Fischer-Marsden Analysis

$L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\mathrm{Ric}(g)$.

As L_g^* is overdetermined-elliptic (injective symbol), it admits a Hodge decomposition/Fredholm alternative:

Hodge-type Decomposition

On closed manifolds (or non-compact manifolds with asymptotic conditions, say) the appropriate function spaces (Sobolev, Hölder, possibly weighted) split as $\mathrm{Im}(L_g) \oplus \ker(L_g^*)$.

Thus by the IFT, we have

Theorem (FM)

Suppose (M^n, g) closed, and $\ker(L_g^*) = \{0\}$. Then there are $\epsilon > 0$ and $C > 0$ so that for all $S \in C^\infty(M)$ with $\|S\|_k < \epsilon$, there is a smooth h with $\|h\|_{k+2} \leq C\|S\|_k$ so that $g + h$ is a Riemannian metric with $R(g+h) = R(g) + S$.

Fischer-Marsden Analysis

$$L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\mathrm{Ric}(g)$$

Corollary

Suppose M^n (closed, connected) does not admit a metric of PSC. Then any metric of nonnegative scalar curvature must be Ricci-flat, and so in three dimensions, the metric must be flat.

Proof: Suppose $R(g) \geq 0$. By the previous theorem, $\ker(L_g^*) \neq \{0\}$, else M would admit PSC. Since M admits a nontrivial static potential f with $L_g^* f = 0$, as we've seen earlier $R(g)$ is constant, thus $R(g) = 0$. From $L_g^* f = 0$, we take the trace to get $-(n-1)\Delta_g f = 0$. So f is harmonic, hence a non-zero constant, WLOG $f = 1$. Thus $0 = L_g^* 1 = -\mathrm{Ric}(g)$.

Localized deformation theorem

- There is a localized analogue of the Fischer-Marsden theorem.

Theorem (Localized deformation of scalar curvature), C.

Suppose $\Omega \subset (M, g)$ is a smooth bounded domain. Suppose that the kernel of L_g^* on Ω is trivial (so (Ω, g) is not static). For $\Omega_0 \Subset \Omega$, there is an $\epsilon_0 > 0$ so that for any $S \in C^\infty(\Omega)$ with $\mathrm{spt}(S) \subset \Omega_0$ and $\|S\|_{C^{0,\alpha}} < \epsilon_0$, there is a smooth metric $g + h$ with $\mathrm{spt}(h) \subset \overline{\Omega}$ with $R(g+h) = R(g) + S$ and $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$.

Remark: Note the nondegeneracy condition about the kernel of L_g^*. For example, Euclidean space admits nontrivial kernel. And note that indeed the conclusion of the Theorem fails in this case: there are no metrics on Euclidean space with $R(g) \geq 0$, $\{R(g) > 0\} \neq \emptyset$, with $g = g_{\mathbb{E}}$ near infinity (no PSC metric on a torus/PMT).

Proof of Local Deformation: Basic estimates

Recall L^2-Sobolev norm $\|u\|^2_{H^k(U)} = \sum_{j=0}^{k} \|\nabla^j u\|^2_{L^2(U)}$.

Basic estimate

The key to obtaining h compactly supported is the form of the operator L_g^*: $L_g^* u = -(\Delta_g u) g + \mathrm{Hess}_g u - u\,\mathrm{Ric}(g)$. One can then get an immediate *a priori* estimate of the form

$$\|u\|_{H^2(U)} \leq C(n, g, \Omega)\left(\|L_g^* u\|_{L^2(U)} + \|u\|_{H^1(U)}\right)$$

valid on any $U \subset \Omega$.

No boundary condition has been imposed.

Proof: $tr_g(L_g^* u) = -(n-1)\Delta_g u - uR(g)$. Thus the full Hessian of u can be expressed in terms of $L_g^* u$ and lower order terms in u.

Proof of Local Deformation: Basic estimates

Coercivity estimate

In case L_g^* has trivial kernel in $H^2_{\mathrm{loc}}(\Omega)$, then one can use compactness arguments to prove the following estimate for small $\epsilon > 0$, where $\Omega_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$:

$$\|u\|_{H^2(\Omega_\epsilon)} \leq C\|L_g^* u\|_{L^2(\Omega_\epsilon)}.$$

Weighted estimates

Let ρ be a smooth, positive function on Ω, which near the boundary is $e^{-1/d}$, $d(x) = \mathrm{dist}(x, \partial\Omega)$. Let $\|u\|^2_{H^k_\rho(\Omega)} = \sum_{j=0}^{k} \|\nabla^j u\|^2_{L^2(\Omega, \rho\, dv_g)}$.

The preceding estimate can be integrated to obtain the weighted L^2 estimates:

$$\|u\|_{H^2_\rho(\Omega)} \leq C\|L_g^* u\|_{L^2_\rho(\Omega)}.$$

Proof of Local Deformation: Linearized problem

Under the assumption that L_g^* has trivial kernel, solutions to $L_g(h) = \sigma$ for $\sigma \in L^2_{\rho^{-1}}(\Omega) = H^0_{\rho^{-1}}(\Omega)$ can be obtained from standard variational arguments. The solution h will be of the form $h = \rho L_g^* u$. The decaying weight ρ will ensure h extends smoothly by zero across $\partial\Omega$.

Proposition

Let $\Omega \subset M$ be a smooth, bounded domain and assume that L_g^* has trivial kernel in $H^2_{\mathrm{loc}}(\Omega)$. Let $\sigma \in L^2_{\rho^{-1}}(\Omega)$. There is a unique minimizer $u \in H^2_\rho(\Omega)$ of the functional $\mathcal{G}$, defined on $H^2_\rho(\Omega)$ given by

$$\mathcal{G}(u) = \int_\Omega \left(\frac{1}{2} \mid L_g^* u \mid^2 \rho - \sigma u \right) d\mu_g .$$

The minimizer is a weak solution of the equation $L_g(\rho L_g^* u) = \sigma$ and satisfies $\|u\|_{H^2_\rho(\Omega)} \leq C \|\sigma\|_{L^2_{\rho^{-1}}(\Omega)}$.

Comments on pointwise estimates

The proof of the Deformation Theorem proceeds through iterated linear corrections. We want to solve $R(g+h) - R(g) - S = 0$. We can solve $L_g(h) = S$, with a norm of h controlled by that of S. For S small enough, $g+h$ is a metric, and so

$$R(g+h) - R(g) = L_g(h) + Q_g(h) = S + Q_g(h).$$

We solved the problem up to quadratic error. Moreover $h = \rho L_g^* u$, and ρ decays near the boundary to all orders.

We iterate the process of linear corrections, and use PDE estimates to keep pointwise control during the iteration—which then converges for S small enough. The solution to the nonlinear problem will be smooth by elliptic theory.

Application: Localized CSC gluing

We saw earlier how to use conformal methods to glue two manifolds of the same CSC together. Suppose these metrics g_1 and g_2 admit trivial kernel for $L^*_{g_i}$ on connected open sets $U_i \ni p_i$.

Lemma

There is $\rho_0 > 0$ so that for $0 < \rho < \rho_0$, $\Delta_{g_i} + \frac{\sigma_n}{n-1}$ has positive Dirichlet spectrum on $B(p_i, \rho)$, and the kernel of $L^*_{g_i}$ is trivial on $U_i \setminus \overline{B(p_i, \rho)}$.

As a corollary we obtain the following.

Corollary (C., Eichmair, Miao)

If (M_1, g_1) and (M_2, g_2) are two n-manifolds of CSC σ_n, with $U_i \subset M_i$ on which $L^*_{g_i}$ has trivial kernel, there is a family of CSC σ_n metrics γ_T on $M_1 \# M_2$ and which are equal to the original metrics on $M_1 \setminus U_1$ and $M_2 \setminus U_2$.

Gluing and asymptotics

The only asymptotically flat metric of nonnegative scalar curvature which is exactly Euclidean near infinity is Euclidean space. What about if we tried to replace the neighborhood of infinity by an exact AF end of a Schwarzschild metric?

Remark: H. Bray proved that you can arrange a neighborhood of infinity to be precisely Schwarzschild provided you allow the nonnegative scalar curvature to be positive in some places.

Question: What if we want to solve the vacuum constraint $R(g) = 0$ and have a neighborhood of infinity be precisely Schwarzschild?

Answer: Yes! We can do it with gluing.

Gluing and asymptotics

In fact we have more than just existence. Given a number m and a vector $c \in \mathbb{R}^n$, we let $g^S_{m,c}(x) = \left(1 + \frac{m}{2|x-c|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathbb{E}}$ be the Schwarzschild metric of mass m and center c.

Theorem (C.)

Let $\mathcal{E} = \{x \in \mathbb{R}^n : |x| > 1\}$. If $(\mathcal{E}, g_0)$ is asymptotically flat with $R(g_0) = 0$ and nonzero ADM mass m_0, then there is an θ_0 so that for $\theta \geq \theta_0$, there is a metric g_θ so that $R(g_\theta) = 0$, $g_\theta = g_0$ for $1 < |x| < \theta$, and g_θ is agrees with a spatial Schwarzschild metric $g^S_{m,c}$ on $|x| > 2\theta$.

Since the deformation is local to an end, the theorem can be applied to an end of any (complete) asymptotically flat manifold with nonzero mass, which by the PMT is any AF with zero scalar curvature which isn't Euclidean space!

This says you can preserve any set with compact closure in $\overline{\mathcal{E}}$, solve the vacuum constraints, and arrange exactly Schwarzschild asymptotics. Since the manifold $\mathcal{E}$ is not complete, $m < 0$ is allowed.

We just briefly outline the proof of the construction: we want to glue on a Schwarzschild AF end to an AF end of of metric g with $R(g) = 0$, keeping zero scalar curvature.

We assume $g_{ij}(x) - \delta_{ij} = O_*(|x|^{-q})$ for $q > \frac{n-2}{2}$. For simplicity, keep in mind the case $n = 3$, $q = 1$.

Gluing and asymptotics

Proof:

• Re-scale the metric g to the unit annulus $1 \leq |x| \leq 2$, to $g^*_\theta = \theta^{-2} f^*_\theta g$, where $f_\theta : A_1 \to A_\theta = \{\theta \leq |x| \leq 2\theta\}$. g^*_θ is close to $g_{\mathbb{E}}$ on the unit annulus, for large θ.

• Produce a metric $\tilde{g}_\theta = \chi g^*_\theta + (1-\chi) g^S_{m/\theta, c}$, so that $\tilde{g}_\theta = g^*_\theta$ in an region around $|x| = 1$, and $\tilde{g}_\theta = g^S_{m/\theta, c}$ in a region around $|x| = 2$, where $|c| < \frac{1}{2}$ and $m \approx m_0$, for θ large. Resulting metric $\tilde{g}_\theta$ is an approximate solution of the constraint: it's close to flat!

• Seek to impose the constraints by localized deformation. Obstruction: Approximate kernel: $L^*_{\tilde{g}_\theta}(1) \approx 0$, $L^*_{\tilde{g}_\theta}(x^i) \approx 0$: coercivity estimate $\|u\|_{H^2} \lesssim \|L^*_{\tilde{g}_\theta} u\|_{L^2}$ not uniform in θ.

• Key point: you do get uniform estimate transverse to this approximate kernel (finite-dimensional). So...

Gluing and asymptotics

• Modify the localized deformation procedure to prove a projected localized deformation theorem for h_θ. This leaves you with $R(\tilde{g}_\theta + h_\theta)$ lying in a finite-dimensional space. You can project this scalar curvature in L^2 onto the basis $\{1, x^1, \ldots, x^n\}$ of $\ker(L^*_{g_{\mathbb{E}}})$.

• Think of a self-adjoint linear operator T on $\mathbb{R}^n$. If T has kernel K, then $\Pi_S \circ T : S \to S$, where $\mathbb{R}^n = K \oplus S$ is an isomorphism. (You can always solve the projected problem.)

• In our setting, the relevant linear operator is essentially $L_\gamma L^*_\gamma$, for $g \approx g_{\mathbb{E}}$. The relevant finite-dimensional space is the $\ker(L^*_{g_{\mathbb{E}}})$ (suitably cut off near the boundary of the annulus), and S is the L^2-orthogonal complement.

• h_θ extends smoothly by 0 outside annulus! So, when we scale back, we will have the original metric inside, and the Schwarzschild metric outside (exactly!!), and zero scalar curvature. We're not quite there yet: how to pick which Schwarzschild to fit on near infinity?

On the asymptotic gluing construction

• Let $\hat{g}_\theta = \tilde{g}_\theta + h_\theta$. We want to fine tune the Schwarzschild mass and center of mass to make $R(\hat{g}_\theta) = 0$.

• At this point, all we have is $R(\tilde{g}_\theta + h_\theta)$ is a linear combination $\{1, x^1, \dots, x^n\}$ (cut off to zero near the boundary of the annulus).

Well: when you project $R(\tilde{g}_\theta + h_\theta) \approx R(\tilde{g}_\theta) + L_{g_\theta}(h_\theta)$ into the approximate kernel, you get, up to scale and error terms, the change in mass Δm across the annulus, and mc! The analysis will dictate the mass of the Schwarzschild, and where to center it!

Indeed, $g_\theta \approx g_{\mathbb{E}}$, and the approximate kernel is the kernel of $L^*_{g_{\mathbb{E}}}$, so projection into the kernel is up to error terms (integrate by parts)

$$\theta\Big(\int_{1 \le |x| \le 2} R(\hat{g}_\theta) dx, \int_{1 \le |x| \le 2} x^i R(\hat{g}_\theta) dx \Big) \sim (m - m_0, mc^i).$$

The parameters may now be chosen by a fixed-point/degree argument.

N-body construction

Suppose we have N AF scalar-flat metrics, of masses $m_1, \ldots, m_N$. Choose a compact region of each metric that you want to preserve. Now suppose we choose an end from each, and we modify it to be Schwarzschild near infinity, keeping the constraint that the scalar curvature is zero, as above.

To be able to build an N-body glued configuration, we just need to build a template metric into which we can put the ends of each metric above, as follows: pick points $c_1, \ldots, c_N$ with suitable disjoint neighborhoods around each. For any $m_1, \ldots, m_N$ with $\sum_{j=1}^{N} m_j \neq 0$, and small enough $\epsilon > 0$, there is a scalar-flat metric on $\mathbb{R}^n \setminus \{c_1, \ldots, c_N\}$ which in a punctured neighborhood of each c_j is just $g^S_{\epsilon m_j, c_j}$, and near infinity is $g^S_{\epsilon m, c}$, where $m \approx \sum_{j=1}^{N} m_j$. If you re-scale this configuration by ϵ^{-1}, the masses become $m_1, \ldots, m_N$ and m, and the centers $\epsilon^{-1} c_j$. (Chruściel-C.-Isenberg)

Remark: There is a completely different sort of N-body configuration established by Carlotto and Schoen.

Some considerations.

• Is there a better functional framework for the localized deformation arguments; what are the best (low) regularity results?

• Some gluing at infinity is available in the asymptotically hyperbolic case (cf. Chruściel-Delay, Cortier). Can these be extended?

• Numerical implementation of the initial data and evolution, for N-body initial data, or for small data Schwarzschild near infinity evolving to model pure gravitational radiation (cf. Doulis-Rinne).

• F. Marques proved the space of AF scalar flat metrics on $\mathbb{R}^3$ is connected. One might ask other questions about the moduli space of solutions: are there results about a path connecting two solutions keeping bounds on asymptotic charges? Are there interesting flows on the moduli space?